Waves on a String

(Elmore & Heald 1.9)
Reflections at a Discontinuity in a Medium

There are two boundary conditions at $x = 0$ are that the string be smooth and continuous:

**Boundary Conditions:**
1) $\eta_{\text{left}} = \eta_{\text{right}}$ at $x = 0$, i.e. the string does not break
2) $\left. \frac{\partial \eta}{\partial x} \right|_{\text{left}} = \left. \frac{\partial \eta}{\partial x} \right|_{\text{right}}$

**Proof**

$F_{\perp \text{right}} = \tau_0 \left. \frac{\partial \eta}{\partial x} \right|_{\text{right}}$

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$F_{\perp \text{total}} = \Delta m$

which must go to 0 as $\Delta m$ goes to 0. Therefore,

$F_{\perp \text{right}} = F_{\perp \text{left}} \Rightarrow \tau_0 \left. \frac{\partial \eta}{\partial x} \right|_{\text{right}} = \tau_0 \left. \frac{\partial \eta}{\partial x} \right|_{\text{left}}$

Now, if $\Delta m \neq 0$, then (HW problem), the second boundary condition is that the total vertical force on the bead = $ma_y$
Let us suppose we have an incident wave, $\eta_i$ from the left traveling to the right:

$$\eta_i = A_i e^{i(k_i x - \omega t)} \quad (x < 0)$$

Then the transmitted wave that passes beyond the $x = 0$ barrier and travels from left to right into medium 2 can be written as:

$$\eta_2 = A_2 e^{i(k_2 x - \omega t)} \quad (x > 0)$$

We can apply both boundary conditions at the interface, $x = 0$. From boundary condition 1:

$$\eta_1 = A_1 e^{i\alpha}$, \quad \eta_2 = A_2 e^{i\omega}$$

Therefore,

$$A_1 = A_2$$

From the second boundary condition, after we take the derivative and then set $x = 0$:

$$k_i A_1 = k_2 A_2$$

The above two results are not both simultaneously possible. Therefore, there must exist a third wave.

$$\eta'_i = B_i e^{i(-k_i x - \omega t)} \quad (x < 0)$$

Which is a wave in medium 1 ($x < 0$) traveling in the opposite direction as $\eta_i$, i.e. from the boundary ($x = 0$) moving to the left. This is the reflected wave.
Applying the two boundary conditions again yield the following two results.

\[ A_1 + B_1 = A_2 \]

and

\[ k_1A_1 - k_1B_1 = k_2A_2 \]

The reflection amplitude coefficient is

\[ R_a \equiv \frac{\text{reflected amplitude}}{\text{incident amplitude}} = \frac{B_1}{A_1} \]

Which may be complex, which would mean there is a phase shift of \( B_1 \) with respect to \( A_1 \).

Let us rearrange the result from the second boundary condition to get

\[ k_1B_1 = k_1A_1 - k_2A_2 = k_1A_1 - k_2(A_1 + B_1) \]

\[ B_1(k_1 + k_2) = A_1(k_1 - k_2) \]

Then the reflection amplitude coefficient becomes the following ratio

\[ R_a = \frac{(k_1 - k_2)}{(k_1 + k_2)} \frac{\left( \frac{\omega}{V_{w1}} - \frac{\omega}{V_{w2}} \right)}{\left( \frac{\omega}{V_{w1}} + \frac{\omega}{V_{w2}} \right)} \]

where \( V_{w1} = \sqrt{\frac{\tau_0}{\mu_1}} \) and \( V_{w2} = \sqrt{\frac{\tau_0}{\mu_2}} \).

Likewise, the transmission amplitude coefficient is

\[ T_a \equiv \frac{\text{transmitted amplitude}}{\text{incident amplitude}} = \frac{A_2}{A_1} = \frac{(k_1 + k_1)}{(k_1 + k_2)} = \frac{2k_1}{(k_1 + k_2)} = \frac{2\frac{\omega}{V_{w1}}}{\left( \frac{\omega}{V_{w1}} + \frac{\omega}{V_{w2}} \right)} \]

Obviously, one check on our generalization is if there were no change in medium at \( x = 0 \) and instead the medium is \( l \) on both sides, then we should have no reflection (since there is no boundary) and full transmission. Let us see: if \( k_1 = k_2 \), then
\[ R_a = 0; \quad T_a = 1 \]

which is what we expected.

**Example:** Suppose \( \mu_2 \gg \mu_1 \) (medium 2 is much denser than 1)

Then \( V_{w2} \ll V_{w1} \) and

\[ R_a \approx -1 \quad T_a \approx 0 \]

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**Power Reflection Coefficient**

We showed that the average power carried by a wave is

\[ P = \frac{1}{2} k \omega \tau_0 |A|^2 \]

where \( \tau_0 \) is the tension and \( A \) the amplitude.

The **Power Reflection Coefficient**, \( R_P \), is

\[ R_P \equiv \frac{\text{Power in reflected wave}}{\text{Power in incident wave}} = \frac{\frac{1}{2} k \omega \tau_0 |B_2|^2}{\frac{1}{2} k \omega \tau_0 |A_1|^2} = \frac{|B_2|^2}{|A_1|^2} \]

and the **Power Transmission Coefficient**, \( T_P \), is

\[ T_P \equiv \frac{\text{Power in transmitted wave}}{\text{Power in incident wave}} = \frac{\frac{1}{2} k \omega \tau_0 |A_2|^2}{\frac{1}{2} k \omega \tau_0 |A_1|^2} = \frac{k_2 |A_2|^2}{k_1 |A_1|^2} \]

Now, suppose we introduce a phase shift, \( \phi \), in the reflected wave

\[ \eta_1 = A_1 e^{i(k_1 x - \omega t)} + B_1 e^{i(k_1 x - \omega t + \phi)} \]

Now what is the reflection amplitude coefficient, \( R_a \)? We may rewrite the above as

\[ \eta_1 = A_1 e^{i(k_1 x - \omega t)} + B_1 e^{i\phi} e^{i(k_1 x - \omega t)} \]
Then, we may plug the coefficients in to the equation for $R_a$ as before.

$$R_a \equiv \frac{\text{reflected amplitude}}{\text{incident amplitude}} = \frac{B_i e^{i\phi}}{A_i}$$

Note, $R_a$ is complex! Its magnitude is

$$|R_a| = \frac{B_i}{A_i}$$