Introduction to Fourier Analysis

Let us consider again the clamped string example:

In the past, we assumed that the wave equation has traveling wave solutions of the form \( f(x \pm ct) \) with a harmonic function as a particular example. Now, let’s generalize, and solve the wave equation by a technique called separation of variables.

The wave equation is

\[
\frac{\partial^2 \eta}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \eta}{\partial t^2}
\]

We assume that \( \eta \) can be written as the product of functions of the independent variables

Let

\[ \eta(x,t) = X(x) \cdot T(t) \]

Function only of \( x \) Function only of \( t \)

Then

\[
T \frac{\partial^2 X}{\partial x^2} = \frac{1}{v^2} X \frac{\partial^2 T}{\partial t^2}
\]

or

\[
\frac{v^2}{X} \frac{\partial^2 X}{\partial x^2} = \frac{1}{T} \frac{\partial^2 T}{\partial t^2}
\]

depends only of \( x \) depends only of \( t \)
This requires that a function of $x$ be equal to a function of $t$ for all values of $x$ and $t$. Since $x$ and $t$ are independent variables, both sides of the equation must equal a constant. Let us call this constant: $-\omega^2$ (separation constant). Then,

$$\frac{v^2 \partial^2 X}{X} = -\omega^2 \frac{\partial^2 X}{\partial x^2}$$

So,

$$\frac{\partial^2 X}{\partial x^2} + \omega^2 X = 0 \quad (1)$$

And similarly,

$$\frac{\partial^2 T}{\partial x^2} + \omega^2 T = 0 \quad (2)$$

You should recognize these two equations. These are associated with the simple harmonic oscillator (SHO).

Let $v = \frac{\omega}{k}$ and then $k = \frac{\omega}{v}$ as usual. Then the solutions of (1) and (2) are

$$X(x) = A \cos(kx) + B \sin(kx)$$

$$T(t) = C \cos(\omega t) + D \sin(\omega t)$$

Proof:

$$\frac{\partial X}{\partial x} = -Ak \sin(kx) + Bk \cos(kx)$$

$$\frac{\partial^2 X}{\partial x^2} = -A k^2 \cos(kx) - B k^2 \sin(kx) = -k^2 X = -\frac{\omega^2}{v^2} X$$

So the most general solution is

$$\eta(x,t) = X(x) \cdot T(t)$$

$$= AC \cos kx \cos \omega t + AD \cos kx \sin \omega t + BC \sin kx \cos \omega t + BD \sin kx \sin \omega t$$

If we choose the left of the string at $x = 0$, then we require
\[ \eta = 0 \quad \text{for} \ x = 0 \ \text{and} \ x = L \quad \text{for all} \ t. \]

Since \( \sin kx = 0 \) and \( \cos kx = 1 \) at \( x = 0 \), this forces \( AC = AD = 0 \) (since \( \cos\omega t \) and \( \sin\omega t \) are generally non-zero). Hence,

\[ \eta(x, t) = BC \sin kx \cos \omega t + BD \sin kx \sin \omega t \]

Since \( \sin kx = 0 \) at \( x = 0 \) and \( x = L \) (for \( \eta = 0 \) at \( x = 0 \)), then \( \sin kl = 0 \) implies \( kl = n \pi \) (\( n \) is an integer). Solving for \( k \), we have:

\[ k_n = \frac{n\pi}{L} = \frac{2\pi}{\lambda_n} \quad \Rightarrow \quad \lambda_n = \frac{2L}{n} \]

Therefore, only certain values of \( \lambda \) are allowed.

But, since \( v = \text{constant} \) and \( \omega = kv \),

\[ \omega_n = k_n v = \frac{n\pi}{L} v \]

Thus,
\[ \eta(x,t) = \sin k_n x (B \cos \omega_n t + D \sin \omega_n t) \]
\[ = \sin k_n x (a_n \cos \omega_n t + b_n \sin \omega_n t) \]

Is one possible mode of oscillation of the string segment. The most general solution of the wave equation consists of an infinite sum of the “normal mode” oscillations. We may write such a solution as

\[ \eta(x,t) = \sum_{n=1}^{\infty} \sin k_n x (a_n \cos \omega_n t + b_n \sin \omega_n t) \]

(This is an example of a Fourier series in both \(x\) and \(t\) – more soon!)

How do we determine the coefficients \(a_n\) and \(b_n\)? We can find them if we know \(\eta(x,0)\) and \(\frac{\partial \eta(x,t)}{\partial t}\) at a particular time.

**Example:**

Let us consider a case where a string is plucked from rest (no initial velocity). Then

\[ \frac{\partial \eta}{\partial t} \bigg|_{t=0} = 0 \quad \text{and} \quad \eta(x,0) = \sum_{n=1}^{\infty} a_n \sin k_n x \]

Let us take the partial time derivative of the most general form of the solution and set it to zero as an initial condition. Therefore, the displacement for any position and time is represented by:

\[ \eta(x,t) = \sum_{n=1}^{\infty} \sin k_n x (a_n \cos \omega_n t + b_n \sin \omega_n t) \]

The velocity for any position and time is:

\[ \frac{\partial \eta(x,t)}{\partial t} = \sum_{n=1}^{\infty} \sin k_n x (-\omega_n a_n \sin \omega_n t + \omega_n b_n \cos \omega_n t) \]

Now, the initial velocity = 0. Hence.
The standard procedure for finding $a_n$ and $b_n$ consist of multiplying each equation by \( \sin k_n x \) and integrating over the length of the string.

From orthogonality

\[
\int_0^L \sin \left( \frac{m \pi x}{L} \right) \sin \left( \frac{n \pi x}{L} \right) \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{2} & \text{if } m = n \end{cases}
\]

It is as if this product is +ive as much as –ive, so the integral = 0. If \( m = n \), then

\[
\int_0^L \sin^2 \left( \frac{n \pi x}{L} \right) \, dx = \frac{L}{2}
\]

In general

\[
\int_0^L \sin \left( \frac{m \pi x}{L} \right) \sin \left( \frac{n \pi x}{L} \right) \, dx = \frac{L}{2} \delta_{mn}
\]

where the kronecker delta, \( \delta_{mn} \), is defined as

\[
\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}
\]

 Aside:

We can do this integral:

\[
\int_0^\pi d \theta \sin^2 n \theta = \frac{\pi}{2},
\]

by using \( \sin \theta = \frac{e^{i \theta} - e^{-i \theta}}{2i} \).

Let us find the \( a_n \) by using “Fourier’s trick”:

We know
\( \eta(x,0) = \sum_{n=1}^{\infty} a_n \sin(k_n x) \)

So we have

\[
\sum_{n=1}^{\infty} a_n \int_0^L \sin\left( \frac{n \pi x}{L} \right) \sin\left( \frac{m \pi x}{L} \right) dx = \int_0^L \eta(x,0) \sin\left( \frac{m \pi x}{L} \right) dx = a_n \frac{L}{2}
\]

integral vanishes except when \( m = n \) where it equals \( L/2 \)

So,

\[
a_n = \frac{2}{L} \int_0^L \eta(x,0) \sin\left( \frac{n \pi x}{L} \right) dx
\]

To find \( b_n \):

Using the relation:

\[
\frac{\partial \eta(x,t)}{\partial t} \bigg|_{t=0} = 0 = \sum_{n=1}^{\infty} b_n \omega_n \sin k_n x
\]

we have

\[
\int_0^L \frac{\partial \eta}{\partial t} \bigg|_{t=0} \sin\left( \frac{m \pi x}{L} \right) dx = \sum_{n=1}^{\infty} b_n \int_0^L \sin\left( \frac{n \pi x}{L} \right) \sin\left( \frac{m \pi x}{L} \right) dx = b_n \frac{L}{2} = 0
\]

integral vanishes except when \( m = n \) where it equals \( L/2 \)

since

\[
\frac{\partial \eta}{\partial t} \bigg|_{t=0} = 0
\]

Therefore,

\[
b_n = 0
\]

To see how this method works out in detail for a particular case, suppose that the string is plucked at center a small distance \( A \) (\( A \ll L \)) and released at time \( t = 0 \). The initial velocity \( \dot{\eta}(x,0) \) is 0.
The initial shape is a triangle (linear) given by the following equation:

\[
\eta(x,0) = \begin{cases} 
\frac{2A}{L} x & 0 \leq x \leq L/2 \\
\frac{2A}{L} (L-x) & L/2 \leq x \leq L 
\end{cases}
\]

This is the initial boundary values (conditions). Note, at \( x = L/2 \), then \( \eta = A \).

Solving for the coefficients \( a_n \) we get:

\[
a_n = \frac{4A}{L} \left[ \int_{0}^{L/2} x \sin \left( \frac{n\pi x}{L} \right) dx + \int_{L/2}^{L} \left( 1 - \frac{x}{L} \right) \sin \left( \frac{n\pi x}{L} \right) dx \right] \\
= 0 \text{ when } n \text{ is even}
\]

When \( n \) is odd, the two integrals are equal and we get

\[
a_n = \frac{8A}{L^2} \int_{0}^{\pi/2} x \sin \left( \frac{n\pi x}{L} \right) dx \\
= \frac{8A}{\pi^2} \int_{0}^{\pi/2} \theta \sin(n\theta) d\theta
\]

where

\[
\theta \equiv \frac{n\pi x}{L}
\]

So

\[
a_n = \frac{8A}{\pi^2} \left( -1 \right)^{(n-1)/2} \frac{1}{n^2} \quad n = 1, 3, 5, \ldots
\]

Hence, we get

\[
\eta(x,t) = \frac{8A}{\pi^2} \left[ \frac{1}{L} \sin \left( \frac{\pi x}{L} \right) \cos(\omega t) - \frac{1}{3^2} \sin \left( \frac{3\pi x}{L} \right) \cos(3\omega t) + \ldots \right]
\]

where
\[ \omega_i = v k_i = \frac{\pi v}{L}. \]