

Fourier Series and Fourier Transforms

1 Fourier Series

A function $f(t)$ which is periodic and “well-behaved”\(^1\) may be represented by a Fourier series:

$$f(t) = a_0/2 + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t))$$

The coefficients $a_n$ and $b_n$ are found from $f(t)$ via

$$a_n = \frac{2}{T} \int_0^T f(t) \cos(n\omega t) dt$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin(n\omega t) dt$$

where $T$ is the period, and $\omega = 2\pi/T$.

The function $f(t)$ is thus identical to a set of harmonic waves, with amplitudes $a_n$ and $b_n$ for the sine and cosine waves of the “$n$th” harmonic. We could also represent the sine and cosine waves as a single harmonic wave of amplitude $c_n$ and phase $\phi_n$; i.e.

$$a_n \cos(n\omega t) + b_n \sin(n\omega t) = c_n \cos(n\omega t + \phi)$$

where

$$c_n = \sqrt{a_n^2 + b_n^2}$$

and

$$\tan \phi = \frac{-b_n}{a_n}$$

2 Complex Fourier Series

There is a complex representation of the Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} d_n e^{in\omega t}$$

where $d_n$ is a complex number, related to the $a_n$ and $b_n$ by

$$d_n = \frac{a_n - ib_n}{2} \quad n \geq 0$$

$$d_n = \frac{a_n + ib_n}{2} \quad n < 0$$

\(^1\)The function must be single-valued; the function and its first derivative must be piecewise continuous.
The magnitude of $d_n$ is
\[ |d_n| = \frac{1}{2} \sqrt{a_n^2 + b_n^2} \]

It is determined from $f(t)$ via
\[ d_n = \frac{1}{T} \int_0^T f(t) e^{-i\omega t} dt \]

As a simple example, consider
\[ f(t) = \sin(\omega t) \]

Then $a_n = 0$ for all $n$, $b_1 = 1$, other $b_n = 0$, $d_1 = -i/2$ and $d_{-1} = +i/2$. As another simple example, consider
\[ f(t) = \cos(\omega t) \]

Then $b_n = 0$ for all $n$, $a_1 = 1$, other $a_n = 0$, $d_1 = +1/2$ and $d_{-1} = +1/2$.

3 Fourier Transform

The frequency spectrum of a periodic function, extending from $-\infty$ to $+\infty$, is discrete. However, the frequency spectrum of "pulse", i.e., a function which is constant or zero everywhere except over a finite time interval, is not discrete. It is represented by a continuous frequency spectrum function $g(\nu)$. The generalization of Equation 1 for this case can be shown to be
\[ f(t) = \int_{-\infty}^{\infty} g(\nu) e^{i2\pi\nu t} d\nu \]

where the frequency spectrum function, $g(\nu)$ is
\[ g(\nu) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi\nu t} dt \]  

(2)

The function $g(\nu)$ is also called the Fourier Transform of $f(t)$. Equation 2 is also written in terms of the angular frequency $\omega$. The above form has the advantage of having no extra normalization constant. Just as the sum in Equation 1 extends from $-\infty$ to $+\infty$ the parameter $\nu$ extents over the same range. For positive values, we may identify $\nu$ with the physical frequency.

3.1 Fourier Transform of a Rectangular Pulse

Let’s find the Fourier transform of a rectangular pulse of total width $T$ and height $A$. Suppose the pulse starts at $-T/2$ and ends at $+T/2$, as shown in Figure 1.

Then, since the function $f(t)$ is zero elsewhere, the integration limits in Equation 2 may be taken to be $-T/2$ and $T/2$, so the Fourier transform becomes
\[ g(\nu) = \int_{-T/2}^{T/2} A e^{-i2\pi\nu t} dt \]

(3)
\[ = \frac{-A}{i2\pi\nu} \left[ e^{-i2\pi\nu \frac{T}{2}} - e^{+i2\pi\nu \frac{T}{2}} \right] \]
\[ = \frac{A \sin(\pi\nu T)}{\pi\nu} \]

(5)
This plotted in the Figure 2 below.

In general $g(\nu)$ is complex; in this particular case it is real. The spread in frequency can be taken to be approximately the distance from the origin to the first zero of $g(\nu)$. The first zero, at $\nu_1$, occurs when the argument of the sine function is $\pi$, i.e.

$$\pi \nu_1 T = \pi$$

or

$$\nu_1 = \frac{1}{T}$$

If we designate the spread in frequency by $\Delta \nu$ and the spread in time of the pulse by $\Delta t$, then the above equation becomes

$$\Delta \nu \Delta t \approx 1$$

This means, generally, that the smaller the pulse width, the larger the range of frequencies it “contains”.

### 3.2 Fourier Transform of a Wave Train

As another example, consider a monochromatic wave train of finite duration:

$$f(t) = \begin{cases} 
  A \cos(2\pi \nu_0 t) & |t| < T \\
  0 & |t| > T 
\end{cases}$$

Figure 3 shows a graph of an example of such a train.

One might, at first, think the Fourier transform of a single-frequency should contain only that frequency, i.e. $g(\nu)$ should be a single peak at $\nu_0$. This is actually not the case, as we will
Figure 2: The Fourier transform of a rectangular pulse. The total pulse width is 1.0 time units,

see by calculating the transform:

\[
g(\nu) = \int_{-T}^{T} A \cos(2\pi \nu_0 t) e^{-i2\pi \nu t} dt
\]

We now “elevate” the cosine term, giving

\[
g(\nu) = \frac{A}{2} \int_{-T}^{T} [e^{i2\pi \nu_0 t} + e^{-i2\pi \nu_0 t}] e^{-i2\pi \nu t} dt
\]

We have already performed this type of integral in the previous example. The answer is:

\[
g(\nu) = AT \left[ \frac{\sin(2\pi (\nu - \nu_0) T)}{2\pi(\nu - \nu_0) T} + \frac{\sin(2\pi (\nu + \nu_0) T)}{2\pi(\nu + \nu_0) T} \right]
\]

This is plotted in Figure 4.

The function \(g(\nu)\) peaks sharply at \(\nu = \pm \nu_0\). It is evident that there are some frequencies present which are near but not exactly equal to \(\pm \nu_0\). Using arguments similar to those made above, the frequency spread, defined as the frequency interval from the peak, at \(\nu_0\), to the nearest zero, is

\[
\Delta \nu = \frac{1}{2T}
\]

where \(2T\) is the total duration of the waveform. Only if the waveform is infinite in duration is the wave “pure”, i.e. has a single frequency in its spectrum.
3.3 Effect of Time Translation

A useful theorem concerns the effect on a Fourier transform of shifting the waveform in time. Suppose a pulse waveform has the form

\[ f(t) = p(t - t_0) \]

Then increasing \( t_0 \) simply “slides” the waveform, to the right, along the \( t \) axis. The Fourier transform of this function can be related to the transform of \( p(t) \). Changing variables from \( t \) to \( z = t - t_0 \), we have

\[
\begin{align*}
    g(\nu) &= \int_{-\infty}^{\infty} p(t - t_0) e^{-i2\pi \nu t} dt \\
    &= \int_{-\infty}^{\infty} p(z) e^{-i2\pi \nu z} e^{-i2\pi \nu t_0} dz \\
    &= g_0(\nu) e^{-i2\pi \nu t_0}
\end{align*}
\]

in which \( g_0(\nu) \) is the Fourier transform of the unshifted pulse. So the effect of a time translation is to make a frequency-dependent phase shift of each Fourier component. However, since the magnitude of the factor in the above equation is 1, the magnitude of the transform is unchanged, i.e.

\[ |g(\nu)| = |g_0(\nu)| \]
Figure 4: The Fourier transform of an harmonic wave train. The frequency is 10 /unit time and the total wave train duration is 2 units of time.

4 Discrete Fourier Transform

The Discrete Fourier Transform (DFT) is used to find the frequency components of a set of sampled values of a waveform. We assume that the waveform $f(t)$ is sampled $N$ times, with the same time interval $\Delta$, between each sample. Then the set of waveform samples, $h(t_k) = h_k$ determines the DFT, $H(\nu_n) = H_n$ through the equation

$$H_n = \Delta \sum_{k=0}^{N-1} h_k e^{-2\pi i k n / N}$$

with the inversion

$$h_k = \frac{1}{N \Delta} \sum_{n=0}^{N-1} H_n e^{2\pi i k n / N}$$

These equations are unlike both the Fourier Series and the Fourier Transform. If, as is frequently the case, the $h_k$ are known, then the set of $H_n$ are completely determined by the above equations. However, the DFT is not an approximating function for representing either $h(t)$ or $H(\nu)$ between the discrete values of $t$ or $\nu$. Note that Equations 13 and 14 do not explicitly contain either $t$ or $\nu$.

The $N$ samples of the function $f(t)$ yield the DFT at $N$ discrete frequencies. They are determined by the Nyquist critical frequency, $\nu_c$:

$$\nu_c = \frac{1}{2\Delta}$$
The frequencies range from \(-\nu_c\) through 0 (DC) to \(+\nu_c\). The ordering of the frequencies with the index \(n\) is given in the following table

<table>
<thead>
<tr>
<th>Index</th>
<th>Frequency</th>
<th>Example: (N = 16, \Delta = 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>(1 \times \frac{1}{N\Delta})</td>
<td>(1/16)</td>
</tr>
<tr>
<td>2</td>
<td>(2 \times \frac{1}{N\Delta})</td>
<td>(2/16)</td>
</tr>
<tr>
<td>(N/2)</td>
<td>(\frac{N}{2N\Delta} = \nu_c, -\nu_c)</td>
<td>(8/16, -8/16)</td>
</tr>
<tr>
<td>(N/2 + 1)</td>
<td>(-\nu_c + \frac{1}{N\Delta})</td>
<td>(-7/16)</td>
</tr>
<tr>
<td>(N/2 + 2)</td>
<td>(-\nu_c + \frac{1}{N\Delta})</td>
<td>(-6/16)</td>
</tr>
<tr>
<td>(N - 1)</td>
<td>(-\nu_c + \frac{N\Delta}{N\Delta})</td>
<td>(-1/16)</td>
</tr>
</tbody>
</table>

In general \(H(\nu_n)\) is complex; it may be reported as separate real and imaginary components, or as a magnitude and a phase. It is also possible for \(h(t_k)\) to be complex.